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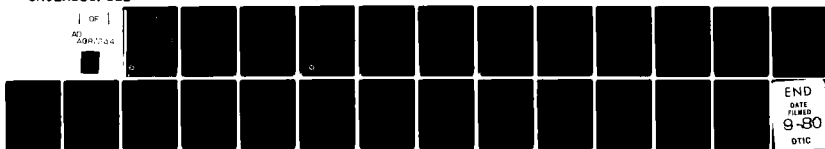
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FRACTURE PROBLEMS IN POWER LAW  
VISCOELASTIC MATERIALS

Arje Nachman

June 1980

Final Report for the period 1 June 1979 - 30 June 1980

Prepared for the  
United States Air Force  
Air Force Office of Scientific Research  
Bolling Air Force Base  
Washington, DC 20332

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  <b>Two papers were completed during the contract year. They are appended to this report. The first has been accepted for publication the International Journal of Solids and Structures. The second has recently been submitted to the same journal. Both papers concern themselves with the general issue of debonding. In each paper loading criteria were exhibited under which an interface edge crack between dissimilar elastic media would propagate. It was felt that these questions must be settled for the elastic case before an intelligent approach to the viscoelastic case could be possible. Moreover, several very interesting</b>			

20. Abstract cont.

viscoelastic debonding problems have recently been formulated by the author, so the general program outlined in the contract will be carried forward.

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## FRACTURE PROBLEMS IN POWER LAW VISCOELASTIC MATERIALS

It is intended to continue the fracture work by examining the Mode III crack in an infinite sheet. Moreover, a very general moduli description will be adopted with the Standard Linear Solid (SLS) and the Power Law (PL) as special cases. The Riemann-Hilbert approach utilized in the original work can be as effectively utilized here. Though the last step of actually producing exact, global closed-form expressions for the field variables is unlikely, it is possible to extract stress intensity factors and near-tip behavior! It should be noted that the Weiner-Hopf approach is inadequate for this problem and that, of course, correspondence principles are inapplicable.

It is also proposed to examine the debonding of an elastic plate on a viscoelastic foundation as a combination of loading, fracture, and lamination. The physical picture would be a load moving along a plate and a semi-infinite crack (whose tip might be in front of the load!) moving with it. The viscoelastic media will be very general again. Coupled field equations for the plate and the halfspace must be solved. The quantities of interest are the position of the crack tip (and its stability in a subsequent analysis), the shape of the crack, and its stress intensity factor. There are also questions regarding the existence of a solution for all load speeds.

Two papers resulted from the research performed under this contract:

- (1) Nachman, A. and Walton, J.R.: Energy Release Rate Calculations for Interface Edge Cracks Based on a Conversation Integral. J. Intern'tl. Solids and Structures, in press.
- (2) Nachman, A. and Walton, J.R.: Energy Release Rate Calculations for an Interface Mode III Edge Crack Based on a

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Conservation Integral. Paper submitted to J. Intern'tl. Solids  
and Structures.

These papers are contained in the Appendix of this report.

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APPENDIX

PAPERS RESULTING FROM  
WORK PERFORMED UNDER  
UNITED STATES AIR FORCE GRANT  
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ENERGY RELEASE RATE CALCULATIONS FOR INTERFACE EDGE CRACKS  
BASED ON A CONSERVATION INTEGRAL

A. Nachman<sup>†</sup> and J. R. Walton<sup>‡</sup>

Introduction

In a recent paper, L. B. Freund [1] demonstrated the usefulness of the M-integral conservation law in the determination of stress intensity factors for 2-dimensional cracks in homogeneous elastic bodies with certain geometric properties. For a detailed discussion of the M-integral and its special features that permit the applications given by Freund, the reader is referred to that paper and the references contained therein. Liberal use will be made in this paper of many of the results derived by Freund.

What is demonstrated here, is that the M-integral can be used to determine the energy release rate for certain interface cracks in much the same way as for cracks in homogeneous bodies.

Three observations are needed for this application. The first, due to Smelzer and Gurtin [2], is that the J-integral on a small arc about an interface crack is equal to the energy release rate, as in the homogeneous case. One difference for cracks in homogeneous bodies is that the J-integral may also be related to the stress intensity factor, while for interface cracks, the J-integral is related to a composite stress intensity factor which has dubious utility. (See Smelzer and Gurtin [2].) The second observation involves the nature of the far stress field for bonded dissimilar elastic wedges, and appeals to the analysis presented by Bogy [3] for bonded dissimilar elastic quarter planes. The third observation is that the integrand of the M-integral is continuous across bonded interfaces lying along radial lines of the chosen coordinate system.

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<sup>†</sup>Department of Mathematics, Old Dominion University, Norfolk, VA. 23508.

<sup>‡</sup>Department of Mathematics, Texas A&M University, College Station, TX 77843

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## Statement and Analysis of the Problem

We present here an extension of the result obtained by Freund for the elastic half-space with an edge crack whose corners are subjected to normal and shear point loads. Specifically, the M-integral conservation law is applied to the 2-dimensional plane strain problem of two infinite isotropic elastic wedges with opening angles  $\omega'$  and  $\omega''$ , respectively, and with different elastic properties (exhibited through  $E'$ ,  $\nu'$  and  $E''$ ,  $\nu''$ , where  $E$  and  $\nu$  denote Young's modulus and Poisson's ratio), which are bonded together along one edge except for a crack of length  $l$  extending from the apex, and whose corners are subjected to a system of normal and shear point loads, given by  $P'$ ,  $Q'$ ,  $P''$  and  $Q''$ . (See Fig. 1.) An application of the M-integral conservation law yields a relation among the parameters  $P$ ,  $Q$ ,  $E$ ,  $\nu$ ,  $l$  and the energy release rate of the crack. (Symbols without primes refer to both materials.) It is clear that the complexity of the problem precludes the determination of the energy release rate by first solving the corresponding boundary value problem.

The M-integral is given by

$$M = \int_C (W n_i x_i - T_k u_{k,i} x_i) ds \quad (1)$$

where  $W$  is the elastic energy density,  $n_i$  is the unit normal to  $C$  (which we take to be directed to the right when  $C$  is traversed in a given direction) and  $T_k$  is the traction acting on the material to the left of  $C$ . The stresses  $\sigma_{ij} = \sigma_{ji}$  are related to the elastic energy  $W$  and the strains  $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$  by  $2W = \sigma_{ij} u_{i,j}$ , where  $u_i$  denote the displacements. The traction  $T_k$  is given by  $T_k = \sigma_{ik} n_i$ . The elastic body is assumed to be in equilibrium without body forces, i.e.  $\sigma_{ij,j} = 0$ , and the stress-strain relation is given by  $\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$ . The conservation law for  $M$  is that  $M = 0$  whenever  $C$  is a closed path surrounding a simply connected region in the body.

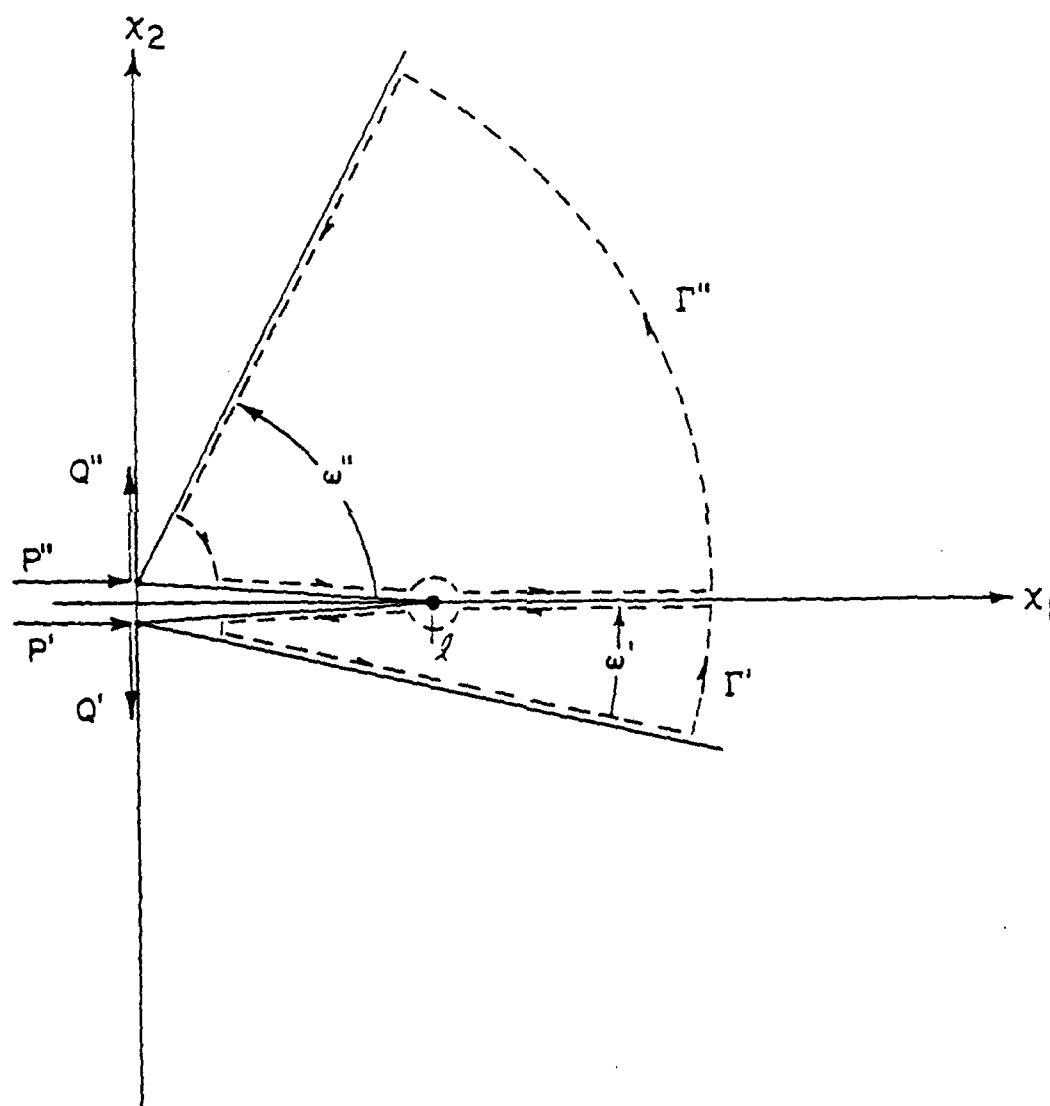


Figure 1.

Consider the integral

$$\begin{aligned}
 M &= M' + M'' \\
 &= \int_{\Gamma} (W n_i x_i - T_k^u u_{k,i} x_i) ds \\
 &= \int_{\Gamma'} (W' n'_i x_i - T'_k u'_{k,i} x_i) ds + \int_{\Gamma''} (W'' n''_i x_i - T''_k u''_{k,i} x_i) ds,
 \end{aligned}$$

where the contours  $\Gamma'$  and  $\Gamma''$  are as indicated in Fig. 1 and  $\Gamma = \Gamma' \cup \Gamma''$ . It follows easily from the discussion by Freund, that there is zero contribution to  $M$  from the parts of  $\Gamma$  along the crack faces and the outside edges of the wedge. Moreover, the value of  $M$  on the vanishingly small arc around the crack tip is  $l \frac{dP}{dl}$ , where  $P(l)$  is the potential energy of the wedge with a crack of length  $l$ . (See Rice [4] and Smelzer and Gurtin [2].) Hence, on this small arc,  $M$  is the product of the crack length and the rate of decrease of the energy with respect to crack length.

The contributions to  $M'$  and  $M''$  from the small arcs around the corner points of the wedge follow from a general result derived by Freund for an infinite elastic wedge with corner loads. In particular, from the small corner arc in  $\Gamma'$  we obtain the contribution

$$\frac{(1 - \nu'^2)}{E'} \left[ \frac{F_a'^2}{\omega' + \sin \omega'} + \frac{F_t'^2}{\omega' - \sin \omega'} \right],$$

and from  $\Gamma''$

$$\frac{(1 - \nu''^2)}{E''} \left[ \frac{F_a''^2}{\omega'' + \sin \omega''} + \frac{F_t''^2}{\omega'' - \sin \omega''} \right],$$

where, employing Freund's notation,

$$F_a' = -P'\cos(\omega'/2) - Q'\sin(\omega'/2),$$

$$F_c' = P'\sin(\omega'/2) - Q'\cos(\omega'/2),$$

$$F_a'' = -P''\cos(\omega''/2) - Q''\cos(\omega''/2),$$

$$F_c'' = -P''\sin(\omega''/2) + Q''\cos(\omega''/2).$$

On the bond line,  $x_2 = 0$ ,  $x_1 > l$ , the contributions to  $M$  from  $\Gamma'$  and  $\Gamma''$  cancel, since, among the stress and displacement components, only  $\sigma_{11}$  is discontinuous across the interface, and on that line  $n_1 x_1 = 0$  (because the interface is radial) and  $T_{k,k,i}^u x_i = (u_{1,1}\sigma_{21} + u_{2,1}\sigma_{12})x_1$ .

It remains to determine the contribution from the large arc for which it suffices to know the far field solution for two wedges bonded together with no crack and with apex loads  $P' + P''$  and  $Q' - Q''$ . From the analysis presented by Bogy [3] for two bonded quarter planes, it is clear that the far field is radial. More precisely, if the elastic fields are represented with respect to polar coordinates  $(r, \theta)$ , then  $r\sigma_{\theta\theta}$  and  $r\sigma_{r\theta}$  vanish for large  $r$  uniformly in  $\theta$ , whereas  $r\sigma_{rr}$  does not. Consequently, on the large arc we may assume  $\sigma_{r\theta}$  and  $\sigma_{\theta\theta}$  are zero. It is easy to see that in this case, to satisfy the equilibrium equations and the compatibility equation, we must take for the stress field

$$\sigma_{rr} = -\frac{A \cos\theta + B \sin\theta}{r}, \quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0. \quad (2)$$

The four constants  $A'$ ,  $B'$ ,  $A''$  and  $B''$  may be calculated from the four conditions

$$u_\theta'(r, \theta \rightarrow 0-) = u_\theta''(r, \theta \rightarrow 0+), \quad (3)$$

$$u_r'(r, \theta \rightarrow 0-) = u_r''(r, \theta \rightarrow 0+), \quad (4)$$

$$\int_{-\omega'}^0 r \sigma'_{rr}(r, \theta) \cos(\theta) d\theta + \int_0^{\omega''} r \sigma''_{rr}(r, \theta) \cos(\theta) d\theta = -(P' + P''), \quad (5)$$

$$\int_{-\omega'}^0 r \sigma'_{rr}(r, \theta) \sin(\theta) d\theta + \int_0^{\omega''} r \sigma''_{rr}(r, \theta) \sin(\theta) d\theta = (Q' - Q''). \quad (6)$$

Equations (3) and (4) assert continuity for the displacements on the bond line, while equations (5) and (6) express the equilibrium of tractions on the wedge  $-\omega' \leq \theta \leq \omega''$ ,  $0 \leq r \leq R$ , where  $R$  is the radius of the large arc. A simple calculation shows that (5) and (6) reduce to

$$A'(1 - \cos 2\omega') - B'(2\omega' - \sin 2\omega') - A''(1 - \cos 2\omega'') - B''(2\omega'' - \sin 2\omega'') = 4(Q' - Q'') \quad (7)$$

$$-A'(2\omega' + \sin 2\omega') + B'(1 - \cos 2\omega') - A''(2\omega'' + \sin 2\omega'') - B''(1 - \cos 2\omega'') = -4(P' + P''). \quad (8)$$

Substitution of (2) into the polar form of the stress-strain law followed by the application of (3) and (4) yields the relations

$$B'/B'' = A'/A'' = (1 + \alpha)/(1 - \alpha), \quad (9)$$

where  $\alpha$  is one of the two Dundurs bi-material parameters given by

(See Bogy [3].)

$$\alpha = \begin{cases} \frac{E'(1 - \nu''^2) - E''(1 - \nu'^2)}{E'(1 - \nu''^2) + E''(1 - \nu'^2)} & \text{for plane strain} \\ \frac{E' - E''}{E' + E''} & \text{for generalized plane stress.} \end{cases}$$

It is now an easy matter to solve equations (7), (8) and (9) for  $A'$ ,  $B'$ ,  $A''$  and  $B''$ . In particular, we obtain

$$A' = (a(Q' - Q'') + b(P' + P''))(4/d)$$

$$B' = (-a(P' + P'') - c(Q' - Q''))(4/d)$$

$$\text{where } a = (1 - \cos 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(1 - \cos 2\omega'')$$

$$b = -(2\omega' - \sin 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(2\omega'' - \sin 2\omega'')$$

$$c = -(2\omega' + \sin 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(2\omega'' + \sin 2\omega'')$$

$$d = a^2 - bc.$$

$A''$  and  $B''$  may now be calculated from (9).

As observed by Freund, for the stress state (2), the integrand of  $M$  is

$$W_{i,n_i} - T_{k,i}^u x_i = -\frac{1}{2} \frac{(1 - v^2)}{E} r \sigma_{rr}^2.$$

Consequently, the contribution to  $M$  from the large arc is

$$\begin{aligned} & -\frac{1}{2} \frac{(1 - v'^2)}{E'} \int_{-\omega'}^0 (A' \cos \theta + B' \sin \theta)^2 d\theta - \frac{1}{2} \frac{(1 - v''^2)}{E''} \int_0^{\omega''} (A'' \cos \theta + B'' \sin \theta) d\theta \\ & = -\frac{1}{2} \frac{(1 - v'^2)}{E'} \left[ \int_{-\omega'}^0 (A' \cos \theta + B' \sin \theta)^2 d\theta + \left(\frac{1-\alpha}{1+\alpha}\right) \int_0^{\omega''} (A' \cos \theta + B' \sin \theta) d\theta \right] \quad (10) \end{aligned}$$

$$= -\frac{(1 - v'^2)}{E'} (b(P' + P'')^2 + c(Q' - Q'')^2 + 2a(P' + P'')(Q' - Q''))(2/d). \quad (11)$$

Line (10) follows from (9) and the observation that

$$\frac{(1 - v''^2)}{E''} = \left(\frac{1+\alpha}{1-\alpha}\right) \frac{(1 - v'^2)}{E'}; \quad \text{whereas, line (11) is derived by simple but tedious algebraic manipulations.}$$

Combining the contributions to  $M$  from the large arc, the two small arcs at the apex and the arc around the crack tip and appealing to the conservation law, we obtain

$$\begin{aligned}
 l \frac{dP}{dl} = & \frac{(1 - \nu'^2)}{E'} [- (b(P' + p'')^2 + c(Q' - Q'')^2 + 2a(P' + P'')(Q' - Q'')) (2/d) \\
 & + \frac{1}{2}(P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1))/(\omega'^2 - \sin^2 \omega') \\
 & + \frac{1}{2}(\frac{1+\alpha}{1-\alpha})(P''^2(2\omega'' - \sin 2\omega'') + Q''^2(2\omega'' + \sin 2\omega'') + 2P''Q''(\cos 2\omega'' - 1))/(\omega''^2 - \sin^2 \omega'')] .
 \end{aligned} \quad (12)$$

An important special case of (12) is that of two bonded dissimilar quarter planes with an edge interface crack. Setting  $\omega' = \omega'' = \pi/2$  in (12) yields

$$\begin{aligned}
 l \frac{dP}{dl} = & \frac{(1 - \nu'^2)}{E'} [(P' + P'')^2 + (Q' - Q'')^2 - \frac{4\alpha}{\pi} (P' + P'')(Q' - Q'')(1 + \alpha)/((2\alpha)^2 - \pi^2) \\
 & + 2((P'^2 + (\frac{1+\alpha}{1-\alpha})P''^2) + (Q'^2 + (\frac{1+\alpha}{1-\alpha})Q''^2) - \frac{4}{\pi} (P'Q' + (\frac{1+\alpha}{1-\alpha})P''Q''))/(\pi^2 - 4)] .
 \end{aligned} \quad (13)$$

It should be noted that if  $P' = P'' \equiv P$ ,  $Q' = Q'' \equiv Q$  and  $\alpha = 0$ , then

(13) reduces to the result obtained by Freund for identical quarter planes.

Of course, when  $\alpha = 0$ , the response of the bonded quarter planes is the same as for a homogeneous half-space; as described by Dundurs [5], the two quarter planes are "consonant in tension parallel to the interface."

Another interesting case in (12) is when  $\alpha = \pm 1$ . Due to symmetry we consider only  $\alpha = -1$  corresponding to which (12) becomes

$$\frac{dP}{dL} = \frac{(1 - \nu'^2)}{2E'} [P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1)] / (\omega'^2 - \sin^2 \omega')$$

It should not be surprising that in (14) only  $P'$  and  $Q'$  appear, since  $\alpha = -1$  corresponds to  $E'' = \infty$ .

It should also be noted that for  $P' = P'' \equiv P$ ,  $Q' = Q'' \equiv Q$  and  $\omega' = \omega'' = \pi/2$ , the level curves of  $L \frac{dP}{dL}$  in the  $(P, Q)$ -plane are ellipses (straight lines if  $\alpha = 0$ ) centered at  $(0, 0)$ . From this we may conclude as did Freund, that crack extension may result when the unloading of  $P$  and  $Q$  occurs along certain paths in the  $(P, Q)$ -plane. Moreover, this obviously is the case also in (14) with  $\omega' \neq \pi/2$ . It is evident in (12), that in general, regardless of the values of  $\omega'$  and  $\omega''$  and  $\alpha$ , this phenomenon is to be expected.

The authors wish to acknowledge with much appreciation several helpful and encouraging discussions on this problem with Professor L. B. Freund.

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ENERGY RELEASE RATE CALCULATIONS FOR AN INTERFACE  
MODE III EDGE CRACK BASED ON A CONSERVATION INTEGRAL

BY

A. Nachman<sup>1</sup> and J. Walton<sup>2</sup>

ABSTRACT

The M-integral is applied to the calculation of energy release rates for interface edge cracks of the Mode III type: specifically, for an edge crack along the interface between two elastic wedges of different opening angles and dissimilar elastic properties, and that is subjected to point loads at the apex, a relation is derived along the length of the crack, the energy release rate of the crack, the applied loads, the wedge angles and the material parameters.

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<sup>1</sup> Department of Mathematical Sciences, Old Dominion University, Norfolk, Virginia 23508. Supported in part by AFOSR Contract F49620-79-C-0076.

<sup>2</sup> Department of Mathematics, Texas A&M University, College Station, Texas 77843. Supported in part by AFORS Grant 77-3290.

In this note we continue the work of [1], wherein the M-integral was used to calculate energy release rates for interface edge cracks of the Mode I and Mode II types. However, here we turn our attention to a Mode III interface edge crack as might arise in the torsion of two welded rods.

The physical scenario is identical to [1] save for the loading at the wedge apexes. In this investigation we impose point shear stresses ( $\sigma_{31}$ ) of magnitudes  $P_1$  and  $P_2$ . See Figure 1. This loading will generate only out-of-plane displacements,  $u_3(x, y)$ . Moreover,  $u_3(x, y)$  is a harmonic function (or, more precisely,  $u_3^{(1)}(x, y)$  and  $u_3^{(2)}(x, y)$  are each harmonic functions in their respective wedges).

We take the same path for the M-integral as in [1]. As before, radial lines contribute nothing, the small circle around the crack tip contributes  $2\frac{dP}{d\ell}$  [2], the interface contributions cancel and we must only deduce the contributions from the arcs near the tips and from the large arc.

Since the edges near each tip are free, an analysis of the near-tip field is equivalent to the analysis of a single-material wedge subject to a shear-stress point load. Clearly the displacement field for such a problem is

$$u_3^{(2)} = A^{(2)} \ln(r) + B^{(2)}, \text{ for the top wedge, say.}$$

The contribution to M from the arc near the top tip is then

$$\int_{w''}^0 r \{ (\sigma_{31}^{(2)} \cos \theta + \sigma_{32}^{(2)} \sin \theta) u_{3,i}^{(2)} x_i - \frac{r}{2} \sigma_{31}^{(2)} u_{3,i}^{(2)} \} d\theta$$

where  $\sigma_{31}^{(2)} = \frac{\mu^{(2)}}{r} [A^{(2)} \cos \theta - B^{(2)} \sin \theta]$  and  $\sigma_{32}^{(2)} = \frac{\mu^{(2)}}{r} [A^{(2)} \sin \theta + B^{(2)} \cos \theta]$ . A simple calculation shows that the integral equals  $\frac{\mu^{(2)} w''}{2} \{ [B^{(2)}]^2 - [A^{(2)}]^2 \}$ . Since  $\int_0^{w''} r \sigma_{rz}^{(2)} d\theta = -P_2 \cos(\frac{w''}{2})$

and  $\int_0^{w''} r \sigma_{\theta z}^{(2)} d\theta = -P_2 \sin(\frac{w''}{2})$  we have that  $A^{(2)} = \frac{-P_2 \cos(\frac{w''}{2})}{2\mu^{(2)} w''}$

and  $B^{(2)} = \frac{-P_2 \sin(\frac{w''}{2})}{2\mu^{(2)} w''}$ . Consequently, the contribution to M

from both near-tip arcs is

$$- \frac{P_2^2}{8\mu^{(2)}} \frac{\cos w''}{w''} - \frac{P_1^2}{8\mu^{(1)}} \frac{\cos w'}{w'} \quad (1)$$

The contribution from the large arc comes from the analysis, as in [1], of the far-field solution for two wedges bonded together with no crack. Here the displacement field in each wedge is likewise of the form  $u_3 = C \ln(r) + D\theta$ .

Since the shear stresses corresponding to the above displacement field are

$$\sigma_{31} = \mu \left( \frac{C}{r} \cos \theta - \frac{D}{r} \sin \theta \right) \quad (2a,b)$$

$$\sigma_{32} = \mu \left( \frac{C}{r} \sin \theta + \frac{D}{r} \cos \theta \right)$$

it follows that continuity of  $u_3$  and  $\sigma_{32}$  at the interface is assured when

$$C^{(1)} = C^{(2)} \quad (3a,b)$$

$$\mu^{(1)} D^{(1)} = \mu^{(2)} D^{(2)}$$

We still need two more equations and these come from, as in [1], a force balance. Thus

$$\int_{-w'}^0 r \sigma_{31}^{(1)} d\theta + \int_0^{w''} r \sigma_{31}^{(2)} d\theta = -(P_1 + P_2) \quad (4a,b)$$

$$\int_{-w'}^0 r \sigma_{32}^{(1)} d\theta + \int_0^{w''} r \sigma_{32}^{(2)} d\theta = 0$$

Using (2a,b) and (3a,b) we get

$$\begin{aligned}
C^{(1)} [u^{(1)} \sin w' + u^{(2)} \sin w''] \\
+ u^{(1)} D^{(1)} [\cos w'' - \cos w'] = -(P_1 + P_2)
\end{aligned}
\tag{5a,b}$$

$$\begin{aligned}
C^{(1)} [u^{(1)} (\cos w' - 1) + u^{(2)} (1 - \cos w'')] \\
+ u^{(1)} D^{(1)} [\sin w' + \sin w''] = 0
\end{aligned}$$

Consequently, the contribution to M from the large arc is

$$\begin{aligned}
-\frac{1}{2} u^{(1)} \int_{-w'}^0 \{ [C^{(1)}]^2 - [D^{(1)}]^2 \} d\theta - \frac{1}{2} u^{(2)} \int_0^{w''} \{ [C^{(2)}]^2 - [D^{(2)}]^2 \} d\theta \\
= -\frac{1}{2} u^{(1)} w' \{ [C^{(1)}]^2 - [D^{(1)}]^2 \} - \frac{1}{2} u^{(2)} w'' \{ [C^{(2)}]^2 - [D^{(2)}]^2 \}.
\end{aligned}$$

We content ourselves with exhibiting the particulars of the case  $w' = w'' = w$ . In this case (5a,b), together with the previous calculations, serves to give

$$\begin{aligned}
\frac{dP}{dl} = & \frac{\cos w}{w^3} \left( \frac{P_1^2}{u^{(1)}} + \frac{P_2^2}{u^{(2)}} \right) \\
& + \frac{w(P_1 + P_2)^2}{2(u^{(1)} + u^{(2)}) \sin^2 w} \left[ 1 - \frac{(1 - \cos w)^2 (u^{(2)} - u^{(1)})^2}{4u^{(1)} u^{(2)} \sin^2 w} \right]
\end{aligned}
\tag{6}$$

We assume w/o l.o.g. that  $u^{(1)} > u^{(2)}$  and then some algebra will show that the level curves of (6) are rotated ellipses. Thus, again, crack extension may result when the unloading of  $P_1$  and  $P_2$  occurs along certain paths in the  $(P_1, P_2)$  plane.

### Bibliography

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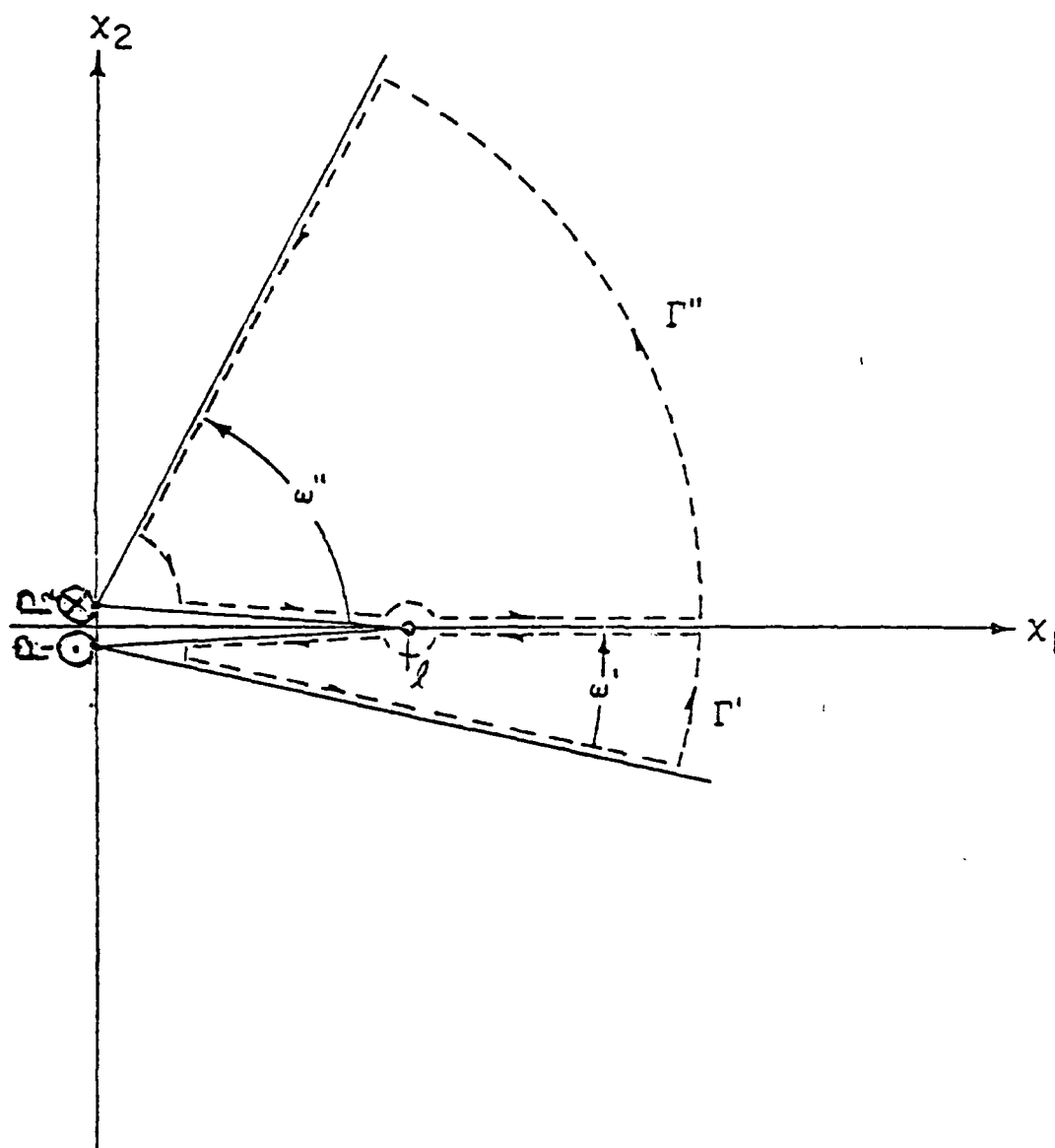


Figure 1.